Interaction of a soliton with point impurities in an inhomogeneous, discrete nonlinear Schrödinger system

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We develop a comprehensive perturbation theory for the inhomogeneous, discrete one-dimensional nonlinear Schrödinger equation based on the inverse scattering transform. We also discuss single-soliton dynamics within the adiabatic approximation and derive higher order corrections to this approximation. Using this perturbation theory, we study in detail the motion of a soliton interacting with a point impurity, either nondissipative or dissipative, in the presence of a spatially linear potential. We predict that there are two kinds of dynamical localization of a soliton in the presence of the nondissipative impurity, depending on the impurity strength. One is the usual dynamical localization, which is qualitatively the same as the one in the absence of the impurity, and the other is the pinning of a soliton by an impurity of sufficient strength. The predictions of these phenomena and their various dynamical properties are confirmed by numerical simulations of the full system. [S1063-651X(96)03005-X]

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I. INTRODUCTION

The dynamics of discrete nonlinear Schrödinger systems has become an important issue in studies of lattice dynamics in condensed matter physics, molecular biology, fiber optics [1,2], etc. As a generalization of a simple tight-binding Schrödinger model to a nonlinear case in the presence of an external electric field, an inhomogeneous, discrete nonlinear Schrödinger equation has been discussed [3–6] in which the nonlinearity arises from interaction of quasiparticles with the lattice (as in the case of excitons in a molecular chain, or electrons in polaronic settings) [3]. It is remarkable that this discrete *nonlinear* system preserves the Bloch oscillations, and, furthermore, exhibits dynamical localization of wave packets [3–6] as in the linear Schrödinger equation with an external, spatially uniform field [7]. It is physically important to realize that these phenomena are consequences of the discreteness of the underlying physical systems of periodic structures, such as in a semiconductor superlattice, an array of coupled quantum wells, or a molecular chain.

In this work, we study effects of point impurities on the dynamics of a discrete soliton in the presence of a spatially linear potential for the above discrete nonlinear Schrödinger system. In particular, we discuss the influence of the presence of the impurity on the phenomenon of dynamical localization for a soliton. To achieve this goal, first we develop a perturbation theory for inhomogeneous, discrete nonlinear Schrödinger equations, based on the inverse scattering transform (IST). Then, we focus on single-soliton dynamics, and study various aspects of its motion within the adiabatic approximation and its higher order corrections. We predict that, depending on the strength of the nondissipative impurities, the dynamics of a soliton exhibits (i) the usual dynamical localization, qualitatively the same as the one in the impurity-free case, and (ii) a trap situation in which the soliton is pinned by an impurity of sufficient strength. In the latter case, the soliton oscillates with a very small spatial amplitude (compared with the localization length of the usual dynamical localization), and is localized around a spatial point which need not be the site of the impurity. Within the adiabatic framework, we estimate the threshold strength above which the pinning of a soliton occurs and the location of the pinning, as well as the correction to the localization length due to the impurity for the usual dynamical localization. These predictions are confirmed by direct numerical simulations of the full dynamical system. For completeness, we also contrast these interesting phenomena induced by a conserved impurity with those induced by a dissipative impurity, for which the total norm of the system is not conserved. We also estimate analytically the change of the amplitude of a soliton after encountering the impurity for two different settings: with or without a static ramp. As we will see below, these theoretical estimates are in good agreement with the numerical simulation results.

The paper is organized as follows: In Sec. II, we present the IST based perturbation theory in detail. Then we study the interaction of a soliton with nondissipative impurities in Sec. III, and with dissipative impurities in Sec. IV. In Sec. V, we conclude the paper.

II. PERTURBATION THEORY FOR INHOMOGENEOUS DNLS EQUATION

The soliton dynamics we deal with is governed by the one-dimensional differential-difference equation

$$i\frac{d\psi_n}{dt}+(\psi_{n+1}+\psi_{n-1})(1+|\psi_n|^2)-2\gamma(t)n\psi_n=\kappa\epsilon\delta_{n,0}\psi_n , \tag{1}$$

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where $\gamma(t)$ is an arbitrary real function of time, and δ_{n0} is the Kronecker δ function localized at the site $n=0$. The positive parameter ϵ characterizes the strength of the impurities. In a perturbative treatment, it is generally assumed that $\epsilon \ll 1$. Here κ signifies the type of impurity: $\kappa=-1$ and $\kappa=1$, corresponding to attractive and repulsive impurities; $\kappa=\pm i$, corresponding to dissipative impurities since, in this case, the norm $\mathcal{N}=\Sigma_n \ln(1+|\psi_n|^2)$ for system (1) is no longer conserved.

Model (1) with $\epsilon=0$ is exactly integrable by means of the inverse scattering transform [3,8]. It is however more convenient to work in terms of the renormalized function [4],

$$q_n(t)=e^{i(2n+1)\Gamma(t)}\psi_n(t), \tag{2}$$

where

$$\Gamma=\int_0^t \gamma(t')dt' \tag{3}$$

since the spectral parameter of the associated linear problem is independent of time [see Eq. (6) below]. The renormalized function $q_n(t)$ satisfies the equation

$$i\frac{dq_n}{dt}+(q_{n+1}e^{-2i\Gamma}+q_{n-1}e^{2i\Gamma})(1+|q_n|^2)+\gamma(t)q_n = \kappa\epsilon\delta_{n,0}q_n . \tag{4}$$

As is shown in Ref. [9], the unperturbed Eqs. (1) and (4) are gauge equivalent.

Following the general ideas in Ref. [10], we generalize the IST based perturbation theory [11] for the Ablowitz-Ladik (AL) model [12] in the following for the discrete nonlinear Schrödinger equation (DNLS) in the form:

$$i\frac{dq_n}{dt}+(q_{n+1}e^{-2i\Gamma}+q_{n-1}e^{2i\Gamma})(1+|q_n|^2)+\gamma(t)q_n = i\epsilon R_n(t), \tag{5}$$

where $R_n(t)$ is a function of n, t, with the boundary conditions that q_n tends to zero sufficiently rapidly as $|n|\to\infty$.

A. Time dependence of the scattering data

As noted above, if $\epsilon=0$ model (5) is integrable by means of the inverse scattering technique. The associated linear spectral problem,

$$F_{n+1}=U_nF_n \tag{6}$$

for the 2×2 matrix function F_n, is defined by the matrix

$$U_n=\begin{pmatrix} z & i\overline{q_n} \\ iq_n & z^{-1}\end{pmatrix}, \tag{7}$$

where z is a spectral parameter (hereafter the bar stands for the complex conjugate). The Jost solutions $T_\pm(n;z)$ of the system (6) are defined by the asymptotics,

$$T_\pm(n;z)\sim\begin{pmatrix} z^n & 0 \\ 0 & z^{-n}\end{pmatrix} \text{ as } n\to\pm\infty. \tag{8}$$

They are related by the transfer matrix $T(z)$,

$$T_-(n;z)=T_+(n;z)T(z). \tag{9}$$

The symmetry of the direct problem (6) implies that the transfer matrix has the following form [12]:

$$T(z)=\begin{pmatrix} a(z) & -\overline{b(z)} \\ b(z) & \overline{a(z)}\end{pmatrix}, \tag{10}$$

where

$$a(z)=\frac{1}{\Delta(n)}\det[T_-^{(1)}(n;z),T_+^{(2)}(n;z)] \tag{11}$$

with

$$\Delta(n)=\det T_+(n;z). \tag{12}$$

The notation $T_\pm^{(j)}(n;z)$ represents the jth column of the matrix $T_\pm(n;z)$.

In order to treat perturbations we need the relation between variations of the transfer matrix and small changes of $U_n(z)$. This is given by the formula

$$\delta T(z)=\sum_{k=-\infty}^{\infty} T_+^{-1}(k+1;z)\,\delta U_k(z)T_-(k;z), \tag{13}$$

which results directly from Eqs. (6) and (9). In particular, it follows from Eq. (13) that

$$\frac{\partial a}{\partial q_n}=-\frac{i}{\Delta(n+1)}T_+^{(12)}(n+1;z)T_-^{(11)}(n;z), \tag{14a}$$

$$\frac{\partial a}{\partial \overline{q_n}}=\frac{i}{\Delta(n+1)}T_+^{(22)}(n+1;z)T_-^{(21)}(n;z), \tag{14b}$$

$$\frac{\partial b}{\partial q_n}=\frac{i}{\Delta(n+1)}T_+^{(11)}(n+1;z)T_-^{(11)}(n;z), \tag{14c}$$

$$\frac{\partial b}{\partial \overline{q_n}}=-\frac{i}{\Delta(n+1)}T_+^{(21)}(n+1;z)T_-^{(21)}(n;z), \tag{14d}$$

where $T_\pm^{(kj)}(n;z)$ denotes the element of the matrix $T_\pm(n;z)$ in the kth row and jth column.

The discrete spectrum of the problem (6) consists of zeros z_k of $a(z)$ outside [or zeros $\tilde{z}_k=\overline{z}_k^{-1}$ of $\overline{a(z)}$ inside] the unit circle [12] and can be parametrized as follows:

$$z_k=e^{w_k+i\theta_k}, \quad \tilde{z}_k=e^{-w_k+i\theta_k}, \tag{15}$$

where w_k is positive, θ_k real, and k labels the zeros.

For z_k in the discrete spectrum one has

$$T_-^{(1)}(n;z_k)=b_kT_+^{(2)}(n;z_k). \tag{16}$$

Taking into account this formula, the relation $a(z_k)=0$, and Eq. (13), one calculates

$$\frac{\partial z_k}{\partial q_n}=\frac{ic_k}{\Delta(n+1)}T_+^{(12)}(n+1;z_k)T_+^{(12)}(n;z_k), \tag{17a}$$

$$\frac{\partial z_k}{\partial \overline{q_n}} = -\frac{ic_k}{\Delta(n+1)} T_+^{(22)}(n+1;z_k) T_+^{(22)}(n;z_k), \quad (17b)$$

$$\frac{\partial b_k}{\partial q_n} = \frac{ib_k}{\dot{a}(z_k)\Delta(n+1)} [\dot{T}_-^{(11)}(n+1;z_k) T_+^{(12)}(n;z_k) - \dot{T}_+^{(12)}(n+1;z_k) T_-^{(11)}(n;z_k)], \quad (17c)$$

$$\frac{\partial b_k}{\partial \overline{q_n}} = \frac{ib_k}{\dot{a}(z_k)\Delta(n+1)} [\dot{T}_+^{(22)}(n+1;z_k) T_-^{(21)}(n;z_k) - \dot{T}_-^{(21)}(n+1;z_k) T_+^{(22)}(n;z_k)], \quad (17d)$$

where $c_k = b_k/\dot{a}(z_k)$ and $\dot{f}(z_k) \equiv df(z)/dz|_{z=z_k}$.

Invoking a similar argument to the one in Ref. [10], we conclude that, if Φ_n is a function of q_n and $\overline{q_n}$, where q_n is governed by the perturbed Eq. (5), then the time dependence of Φ_n is determined by

$$\frac{d\Phi_n}{dt} = \hat{L}\Phi_n + \epsilon \sum_{n=-\infty}^{\infty} \left(\frac{\partial \Phi_n}{\partial q_n} R_n + \frac{\partial \Phi_n}{\partial \overline{q_n}} \overline{R_n} \right), \quad (18)$$

where the operator $\hat{L}$ defines the same dependence on q_n and $\overline{q_n}$ as it would at $\epsilon = 0$. Using the results of Ref. [4] related to the time dependence of the unperturbed scattering data and letting Φ_n be one of the scattering data, one arrives at the following set of equations:

$$\frac{\partial a}{\partial t} = i\epsilon \sum_{n=-\infty}^{\infty} \frac{1}{\Delta(n+1)} [T_+^{(22)}(n+1;z) T_-^{(21)}(n;z) \overline{R_n} - T_+^{(12)}(n+1;z) T_-^{(11)}(n;z) R_n], \quad (19)$$

$$\frac{\partial b}{\partial t} = i\alpha(z,t) b + i\epsilon \sum_{n=-\infty}^{\infty} \frac{1}{\Delta(n+1)} [T_+^{(11)}(n+1;z) T_-^{(11)}(n;z) R_n - T_+^{(21)}(n+1;z) T_-^{(21)}(n;z) \overline{R_n}], \quad (20)$$

$$\frac{dz_k}{dt} = i\epsilon \frac{1}{\dot{a}(z_k)} \sum_{n=-\infty}^{\infty} \frac{1}{\Delta(n+1)} [T_+^{(12)}(n+1;z_k) T_-^{(11)}(n;z_k) R_n - T_+^{(22)}(n+1;z_k) T_-^{(21)}(n;z_k) \overline{R_n}], \quad (21)$$

$$\frac{db_k}{dt} = i\alpha(z_k,t) b_k + i\epsilon \frac{b_k}{\dot{a}(z_k)} \sum_{n=-\infty}^{\infty} \frac{1}{\Delta(n+1)} \{ [\dot{T}_-^{(11)}(n+1;z_k) T_+^{(12)}(n;z_k) - \dot{T}_+^{(12)}(n+1;z_k) T_-^{(11)}(n;z_k)] R_n + [\dot{T}_+^{(22)}(n+1;z_k) T_-^{(21)}(n;z_k) - \dot{T}_-^{(21)}(n+1;z_k) T_+^{(22)}(n;z_k)] \overline{R_n} \}, \quad (22)$$

where

$$\alpha(z,t) = z^2 e^{2i\Gamma} + z^{-2} e^{-2i\Gamma} + \gamma(t). \quad (23)$$

B. Adiabatic approximation

It can readily be seen that Eqs. (19)–(22) have a convenient form for expansions with respect to ϵ. In what follows, we concentrate on one-soliton dynamics. The soliton of the system (5) can be represented as

$$q_n(t) = q_0(n,t) + \epsilon q_1(n,t), \quad (24)$$

where

$$q_0(t) = -i \frac{\sinh(2w) e^{-2i\theta(n-\zeta)+i\nu}}{\cosh[2w(n-\zeta)]} \quad (25)$$

is the so-called adiabatic term (it coincides with the exact one-soliton solution at $\epsilon = 0$ [8]) if one is only interested in the time dependence of the parameters, w, ζ, θ, ν, while keeping the functional form of the soliton (25) fixed. We note that the parameters of the problem are linked by

$$b_1 = z_1 e^{2i\theta\zeta + 2w(\zeta+1) + i\nu}. \quad (26)$$

The Jost matrices of the one-soliton solution are given by

$$T_-^{(1)}(n;z) = \frac{z^n}{2\cosh[2w(n-1-\zeta)]} \times \begin{pmatrix} a_s(z) e^{2w(n-1-\zeta)} + e^{-2w(n-1-\zeta)} \\ [1 - a_s(z)] z e^{-2i\theta(n-\zeta)+i\nu} \end{pmatrix}, \quad (27a)$$

$$T_+^{(2)}(n;z) = \frac{z^{-n}}{2\cosh[2w(n-1-\zeta)]} \times \begin{pmatrix} [1 - e^{-4w} a_s(z)] z^{-1} e^{2i\theta(n-\zeta)-i\nu} \\ e^{2w(n-1-\zeta)} + a_s(z) e^{-2w(n+1-\zeta)} \end{pmatrix}, \quad (27b)$$

where

$$a_s(z) = \frac{z^2 - z_1^2}{z^2 - \tilde{z}_1^2} \quad (28)$$

is the Jost coefficient of the one-soliton solution and z belongs to the unit circle.

As usual, the perturbation analysis implies expansion with respect to the small parameter ϵ. Therefore to work in the first order of ϵ it is sufficient to use the Jost functions and

Jost coefficients associated with the unperturbed soliton solution. Then direct algebra leads to the reduction of the complex Eq. (21) to two real ones,

$$\frac{dw}{dt}=-\frac{\epsilon}{2}\sinh(2w)\sum_{n=-\infty}^{\infty}\frac{\cosh[2w(n-\zeta)]}{\cosh[2w(n+1-\zeta)]\cosh[2w(n-1-\zeta)]}R_n'', \tag{29}$$

$$\frac{d\theta}{dt}=-\frac{\epsilon}{2}\sinh(2w)\sum_{n=-\infty}^{\infty}\frac{\sinh[2w(n-\zeta)]}{\cosh[2w(n+1-\zeta)]\cosh[2w(n-1-\zeta)]}R_n'. \tag{30}$$

Here for the sake of convenience we introduce the notations,

$$R_n'=\operatorname{Re}\{R_n(t)e^{2i\theta(n-\zeta)-i\nu}\}, \tag{31a}$$

$$R_n''=\operatorname{Im}\{R_n(t)e^{2i\theta(n-\zeta)-i\nu}\}. \tag{31b}$$

From Eq. (22) other equations of the adiabatic approximation result, such as the time evolutions for the parameters ζ and ν

$$\frac{d\zeta}{dt}=-\frac{\sinh(2w)}{w}\sin[2(\Gamma+\theta)]-\frac{\epsilon}{2}\frac{\sinh(2w)}{w}\sum_{n=-\infty}^{\infty}\frac{(n-\zeta)\cosh[2w(n-\zeta)]}{\cosh[2w(n+1-\zeta)]\cosh[2w(n-1-\zeta)]}R_n'', \tag{32}$$

$$\frac{d\nu}{dt}=\gamma+2\cosh(2w)\cos[2(\Gamma+\theta)]+2\frac{\theta}{w}\sinh(2w)\sin[2(\Gamma+\theta)]+\frac{\epsilon}{2}\sum_{n=-\infty}^{\infty}\frac{1}{\cosh[2w(n+1-\zeta)]\cosh[2w(n-1-\zeta)]}$$
$$\times\left\{2\cosh[2w(n+1-\zeta)]R_n'-(n+1-\zeta)\sinh(2w)\sinh[2w(n-\zeta)]R_n'+2\frac{\theta}{w}(n-\zeta)\sinh(2w)\cosh[2w(n-\zeta)]R_n''\right\}. \tag{33}$$

In the case of the AL model, i.e., $\gamma\equiv\Gamma\equiv0$, Eqs. (29), (30), and (32) reduce to those obtained in Ref. [11].

C. First order approximation

In order to calculate the first order correction $q_1(n,t)$ introduced in Eq. (24) we recall the equations of the inverse problem obtained by Ablowitz and Ladik [12]. Namely, the ''potential'' in the spectral problem [Eqs. (6) and (7)] is reconstructed from the scattering data by the relation,

$$q_n(t)=-i\overline{K(n,n+1;t)}, \tag{34}$$

where the function $K(n,m;t)$ is found from

$$K(n,m;t)-2F(n+m;t)+4\sum_{p=0}^{\infty}\sum_{q=1}^{\infty}K(n,n+2p+1;t)$$
$$\times\overline{F(2(n+p+q)+1;t)}F(n+m+2q;t)=0, \tag{35}$$

where

$$F(n;t)=F_0(n;t)+F_1(n;t), \tag{36}$$

$$F_0(n;t)=-\overline{c_1}\tilde{z}_1^{n+1}, \tag{37}$$

$$F_1(n;t)=\frac{1}{2\pi i}\oint_R\frac{\overline{b(z;t)}}{\overline{a(z;t)}}z^{n-1}dz, \tag{38}$$

and the integral in Eq. (38) is performed along the right portion of the unit circle (this is indicated by the index R). In the above representation we consider the perturbed dynamics of one soliton only. This restriction implies also that $F_1(n;t)$ is small compared with $F_0(n;t)$. Hence the solution of Eq. (35) can be found in the form (see Ref. [10] for the calculation approach)

$$K(n,m;t)=K_0(n,m;t)+\delta K(n,m;t), \tag{39}$$

where $K_0(n,m;t)$ is a ''pure'' soliton part and $\delta K(n,m;t)$ is a small correction. Naturally,

$$q_0(n,t)=-i\overline{K_0}(n,n+1;t),\quad \epsilon q_1(n,t)=-i\overline{\delta K}(n,n+1;t). \tag{40}$$

Inserting Eq. (39) into Eq. (35) and keeping the terms to the first order in ϵ, one arrives at the equation for $\delta K(n,m;t)$,

$$\delta K(n,m;t)+4\sum_{p=0}^{\infty}\sum_{q=1}^{\infty}\delta K(n,n+2p+1;t)\overline{F_0}(2(n+p+q)$$
$$+1;t)F_0(n+m+2q;t)=B(n,m;t), \tag{41}$$

where

$$B(n,m;t)=2F_1(n+m;t)-4\sum_{p=0}^{\infty}\sum_{q=1}^{\infty}K_0(n,n+2p+1;t)$$
$$\times[\overline{F_1}(2(n+p+q)+1;t)F_0(n+m+2q;t)$$
$$+\overline{F_0}(2(n+p+q)+1;t)F_1(n+m+2q;t)]. \tag{42}$$

By direct substitution one can verify that Eq. (41) is solved by

$$\delta K(n,m;t)=B(n,m;t)+c_1\frac{e^{-2w}}{\sinh(2w)}K_0(n,m;t)\sum_{p=1}^{\infty} B(n,n+2p-1;t)z_1^{-2(n+p)}. \tag{43}$$

Hence with the help of Eqs. (25), (26), (28), and (40) we obtain the representation

$$i\epsilon q_1(n,t)=\overline{B(n,n+1;t)}+\frac{\sinh(2w)e^{-2w(n-\zeta+1)}}{\cosh[2w(n-\zeta)]}\times\sum_{p=0}^{\infty}\overline{B(n,n+1+2p;t)}e^{2ip\theta-2pw}. \tag{44}$$

To find the final expression for $q_1(n,t)$, we have to express $B(n,m;t)$ explicitly in terms of the perturbation function R_n. First, we notice that within the limit of the desired accuracy, $a(z)$ in Eq. (38) can be replaced by $a_s(z)$ defined in Eq. (28). Furthermore, we have

$$T_+^{(1)}(n;z)=\frac{1}{a_s(z)}T_-^{(1)}(n;z)+O(\epsilon). \tag{45}$$

Then, taking into account the explicit form of the Jost function (27a), we can reduce Eq. (20) to

$$\frac{\partial b}{\partial t}=i\alpha(z,t)b+2i\frac{\epsilon a_s(z)e^{2w+4i\theta}}{(z^2-z_1^2)^2}\sum_{n=-\infty}^{\infty}\frac{z^{2n+1}e^{-2i\theta(n-\zeta)+i\nu}}{\cosh[2w(n-\zeta+1)]\cosh[2w(n-\zeta-1)]}\{(R_n'+iR_n'')\{z^2e^{-2i\theta}\cosh[2w(n-\zeta)]-\cosh[2w(n-\zeta+1)]\}\{z^2e^{-2i\theta}\cosh[2w(n-\zeta-1)]-\cosh[2w(n-\zeta)]\}-\sinh^2(2w)(R_n'-iR_n'')\}. \tag{46}$$

This equation can readily be solved for b. Via Eqs. (38) and (42), one arrives at an explicit form of the first order correction (not shown here because of its rather cumbersome form).

III. INTERACTION OF A SOLITON WITH NONDISSIPATIVE IMPURITIES

We have now assembled all necessary machinery to consider the interaction of a soliton with point impurities. Let us start with the case of a single nondissipative impurity $\kappa=\pm 1$ placed at the site $n=0$. In this case, $R_n(t)=-i\kappa\delta_{n,0}q_n$, and the equations of the adiabatic approximation reduce to

$$\frac{dw}{dt}=0, \tag{47}$$

$$\frac{d\theta}{dt}=-\frac{\epsilon}{2}\kappa\sinh^2(2w)\frac{\tanh[2w\zeta]}{\cosh[2w(1-\zeta)]\cosh[2w(1+\zeta)]}, \tag{48}$$

$$\frac{d\zeta}{dt}=-\frac{\sinh(2w)}{w}\sin[2(\Gamma+\theta)], \tag{49}$$

$$\frac{d\nu}{dt}=\gamma+2\cosh(2w)\cos[2(\Gamma+\theta)]+2\frac{\theta}{w}\sinh(2w)\sin[2(\Gamma+\theta)]-\frac{\epsilon}{2}\kappa\frac{\sinh(2w)}{\cosh(2w\zeta)\cosh[2w(1-\zeta)]\cosh[2w(1+\zeta)]}\times\{2\cosh[2w(1-\zeta)]+(1-\zeta)\sinh(2w)\sinh(2w\zeta)\}. \tag{50}$$

As expected, evolutions of the parameters, w, ζ, θ, are not coupled to the evolution of the phase ν, and ν is merely slaved to the dynamics of these parameters. Therefore we only have to deal with Eqs. (48) and (49) describing evolutions of the angle parameters θ and the center ζ of the soliton (according to the terminology adopted in Sec. II, they can be regarded as collective coordinates of the soliton). Note that Eq. (47) is a consequence of the conservation of the norm [see Eq. (52) below].

Before going into details, we point out that, when $R_n''\equiv 0$, as in the case discussed above, the perturbed system still has an integral of motion [3,15]

$$\mathcal{N}=\sum_{n=-\infty}^{\infty}\ln(1+|q_n|^2), \tag{51}$$

which can be viewed as the norm of the system (1), and we have

$$\sum_{n=-\infty}^{\infty}\ln(1+|q_0(n,t)|^2)=4w. \tag{52}$$

Another quantity,

$$C=\sum_{n=-\infty}^{\infty}\overline{q_{n-1}}q_n, \tag{53}$$

which is an integral of motion of the unperturbed model, now evolves in time according to the relation

$$\frac{dC}{dt}=i\epsilon\sum_{n=-\infty}^{\infty}(\overline{q_{n-1}}R_n+\overline{R_{n-1}}q_n). \tag{54}$$

A. Interaction of an AL soliton with a point impurity

If $\gamma\equiv\Gamma\equiv 0$, the system under consideration reduces to the AL model. This situation was studied in Ref. [15]. We describe it from the viewpoint of the IST perturbation theory below.

It follows from Eqs. (48) and (49) that in leading order the collective coordinate of the soliton solves the Newton equation

$$\frac{d^2\zeta}{dt^2}=-\cos(2\theta_0)U'(\zeta), \tag{55}$$

where $U'(\zeta)=dU(\zeta)/d\zeta$, the ''potential'' $U(\zeta)$ is given by

$$U(\zeta)=\epsilon\kappa\frac{\sinh(2w)}{2w}\ln\left(1+\frac{\sinh^2(2w)}{\cosh^2(2w\zeta)}\right), \tag{56}$$

and θ_0 is the initial value of the phase θ.

B. Interaction of a soliton with an impurity in the presence of a constant field

The system of Eqs. (48) and (49) has two time scales: the rapid, conventional time t defines the change of ζ, and a slow time ϵt is related to the variation of the angle variable θ. We exploit this fact below in the treatment of the perturbed soliton dynamics in the presence of a constant field $\gamma=$const and $\Gamma\equiv\gamma t$ (without loss of generality, γ will be assumed to be positive). In the unperturbed case, i.e., $\epsilon=0$, the motion of a soliton is periodic and is described by the solution of Eq. (49) [5,6],

$$\zeta=\frac{\sinh(2w)}{2\gamma w}\cos(2\gamma t+2\theta)+\zeta_1, \tag{57}$$

where $\theta=$const and ζ_1 is a constant related to the initial position of the soliton. (It is interesting to mention here that a dark soliton of the AL model also undergoes periodic motions in a linear field [14].)

Following the general ideas of multiscale expansion (see, e.g., Ref. [13]), for the perturbed case we regard θ and ζ_1 as functions of the slow time. Differentiating Eq. (57) with respect to t and taking into account the fact that the temporal dependence of θ is governed by Eq. (48), we arrive at the relation,

$$\frac{d\zeta_1}{dt}=\frac{\sin(2\gamma t+2\theta)}{4\gamma w}U'(\zeta), \tag{58}$$

[$U(\zeta)$ is given by Eq. (56)]. Within the accepted accuracy, $O(\epsilon^2)$, ζ_1 is evaluated explicitly. Indeed, it follows from Eqs. (49) and (58) that

$$\frac{d\zeta_1}{d\zeta}=-\frac{U'(\zeta)}{4\gamma\sinh(2w)}. \tag{59}$$

Hence

$$\zeta_1=-\frac{U(\zeta)}{4\gamma\sinh(2w)}+\zeta_1^{(0)}, \tag{60}$$

where $\zeta_1^{(0)}$ is a constant determined through initial conditions.

Since the motion is quasiperiodic it is natural to analyze the shift of the center of the soliton oscillations averaged over one period $T=\pi/\gamma$ of the rapid motion. The shift is defined by

$$\Delta\zeta=\zeta(t+T)-\zeta(t). \tag{61}$$

An explicit expression for $\Delta\zeta$ results directly from Eq. (49),

$$\Delta\zeta=-\frac{2\sinh(2w)}{w}\sin[\theta(t+T)-\theta(t)]\cos[2\gamma t+\theta(t+T)+\theta(t)]. \tag{62}$$

Then from Eq. (48) we find that the change of $\theta(t)$ during one period is of the order of ϵ^2. Hence $\Delta\zeta=O(\epsilon^2)$. For similar arguments we can show that the change of the period is also of order $O(\epsilon^2)$.

It is not difficult to calculate the variation of the amplitude of the oscillation. The change of ζ during a half-period is given by

$$\Delta\zeta_{1/2}=\frac{\kappa\epsilon}{8w\gamma}\ln\frac{\cosh[2w(\zeta_{\min}-1)]\cosh[2w(\zeta_{\min}+1)]\cosh^2(2w\zeta_{\max})}{\cosh[2w(\zeta_{\max}-1)]\cosh[2w(\zeta_{\max}+1)]\cosh^2(2w\zeta_{\min})}, \tag{63}$$

where

$$\Delta\zeta_{1/2}=\int_{\zeta_{\min}}^{\zeta_{\max}}d\zeta_1=\int_{\zeta_{\min}}^{\zeta_{\max}}d\zeta-\zeta_{\text{unperturbed}}\Big|_{\zeta_{\min}}^{\zeta_{\max}}, \tag{64}$$

where $\zeta_{\min}=\zeta_1-\sinh(2w)/(2\gamma w)$, $\zeta_{\max}=\zeta_1+\sinh(2w)/(2\gamma w)$ are the minimal and maximal values of the unperturbed soliton trajectory, respectively. It is a direct consequence of this result that $\Delta\zeta_{1/2}>0$ if $\kappa\zeta_1>0$ and $\Delta\zeta_{1/2}<0$ if $\kappa\zeta_1<0$.

A striking property of Eqs. (48) and (49) of the adiabatic approximation is that they allow for a trapped soliton solution. By trapping, we mean mathematically that the right-hand side of Eq. (49) vanishes. This condition has two consequences: (i) $\zeta=\zeta_0=\text{const}$, and (ii) $\theta=-\Gamma(t)$. [Here we temporarily return to the general case in which γ is an arbitrary function of time.] It follows from Eq. (48) that these two conditions are compatible only when $\Gamma(t)=\gamma t$, $\gamma=$ const. Hence solitons cannot be trapped, in general, by the impurity if the external field depends on time. However, for a nonzero constant γ, trapping can take place. Furthermore, we note that, in contrast to the case considered in the previous subsection, now the position ζ_0 of the trapped soliton does not coincide with the position of the point impurity. Indeed, it is defined by the relation,

$$U'(\zeta_0)=-4\gamma\sinh(2w), \tag{65}$$

which follows from Eqs. (48), (56) and the above requirement (ii).

To examine the stability of a soliton placed at ζ_0 we represent $\zeta=\zeta_0+\xi$ with $|\xi|\ll|\zeta_0|$ and deduce from Eqs. (48) and (49) an equation for ξ

$$\frac{d^2\xi}{dt^2}+\frac{1}{2w}U''(\zeta_0)\xi=0. \tag{66}$$

It can immediately be seen from Eq. (66) that the position ζ_0 is stable only if $U''(\zeta_0)>0$. This enables us to find a threshold value ϵ_{th} above which trapping occurs. Obviously, such a threshold corresponds to $U''(\zeta_0)=0$, where ζ_0 is considered as a function of ϵ determined by (65). Direct calculation yields

$$\epsilon_{\text{th}}=\frac{\gamma}{\sqrt{2}\sinh^2(2w)}\,\frac{[1+2\cosh(4w)+\sqrt{5+4\cosh(4w)}]^{3/2}}{\sqrt{5+4\cosh(4w)}-1}, \tag{67}$$

and the value of ϵ_{th} turns out to be the same for both the attractive and the repulsive impurities. For $\epsilon<\epsilon_{\text{th}}$ there is no trapping for either $\kappa=1$ or $\kappa=-1$. But the trapping scenario is different for these two cases. For $\kappa=-1$, the trap position $\zeta_0\to 0$ as $\epsilon\to\infty$ (i.e., the center of the soliton tends to the location of the impurity). In general $\zeta_0\in(0,\zeta_{0,\text{th}})$, where $\zeta_{0,\text{th}}$ satisfies both Eq. (65) and $U''(\zeta_{0,\text{th}})=0$. In contrast, for $\kappa=1$, the trap position $\zeta_0\geqslant\zeta_{0,\text{th}}$ as ϵ increases from ϵ_{th}.

We have performed numerical simulations for the full dynamical system (1) of sufficiently large lattice sizes to avoid any boundary effects. In numerical simulations, we used a Simpson interpolation scheme to find the location of the maximum of the modulus $|q_n(t)|$ as a function of time t. More specifically, we first locate three greatest values, $|q_{m-1}(t)|$, $|q_m(t)|$, and $|q_{m+1}(t)|$, of the moduli, then interpolate parabolically these three points to find the position of the maximum. The implementation of the interpolation enables us to follow a *continuous* change (not limited to the lattice spacing) of the location ζ of the soliton despite the fact that the dynamical system (1) is a lattice model. We summarize our numerical simulation results as follows:

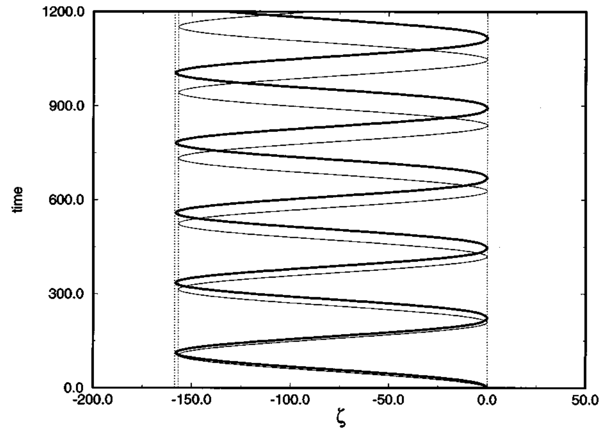

FIG. 1. Trajectories of a soliton [Eq. (68)] with $w=0.5$, $\theta=0$ interacting with an impurity $\epsilon\kappa\delta_{0,n}$ localized at $\zeta=0$ in the presence of a ramp field $V_n=0.03n$. Thick line: $\epsilon\kappa=-0.1$. For comparison, thin line: $\epsilon\kappa=0$, i.e., the unperturbed case. Dotted lines indicate the positions of the turning points of the trajectories. (See text for details.)

(a) For the case of a single site impurity with $\epsilon\kappa=-0.1$ localized at $n=0$ in addition to a linear potential $2\gamma n$ with $\gamma=0.015$, a soliton,

$$q_0(n,t)=\sinh(2w)\text{sech}[2w(n-\zeta)]e^{-2i\theta(n-\zeta)+i\nu}, \tag{68}$$

initially localized at $\zeta=0$ with $w=0.5$, and $\theta=0$ evolves with a perturbed trajectory. Figure 1 shows the trajectory (thick line) of the soliton which is the locus of the interpolated maximum of $|q_n(t)|$ compared with the unperturbed trajectory (thin line). It clearly shows that the amplitude of the oscillatory motion is modified by the presence of the impurity, as is the oscillation frequency. We define the localization length A as the distance between the left turning points and the right turning points of an oscillatory trajectory (see Fig. 1). This is twice the amplitude of the oscillatory motion. We numerically measured that $\Delta A=A_{\text{perturbed}}-A_{\text{unperturbed}}=1.30$ for $\epsilon\kappa=-0.1$, $\Delta A=-1.64$ for $\epsilon\kappa=0.1$, respectively. From Eq. (63) the theoretical estimates for the correction ΔA of the localization length are $\Delta\zeta_{1/2}=\pm 1.46$ for $\epsilon\kappa=\pm 0.1$. Therefore the theoretical estimates and the numerical results agree with each other rather well (with a relative error of about 10%). We note that, in order to measure this small correction in the change of the amplitude, the impurity should be placed around the turning points since the correction $\Delta\zeta_{1/2}$ decays exponentially with a decay width of the order of $1/(2w)$ as the distance increases between the impurity and one of the turning points.

(b) Next we discuss the phenomenon of trapping a soliton by an impurity in the presence of a spatially linear potential. First, we note that Eq. (65) in general has two solutions for

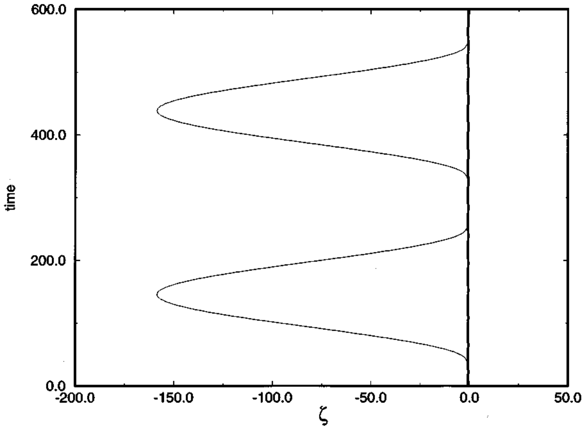

FIG. 2. Untrapped soliton vs soliton pinned by an impurity located at $\zeta=0$ in the presence of $V_n=0.03n$. Thick line shows the trajectory of a trapped soliton executing small amplitude oscillations, here $\epsilon\kappa=-0.1036$. Thin line is the trajectory of an untrapped soliton, $\epsilon\kappa=-0.1010$. The soliton is initially located at $\zeta=-0.8328$ with $w=0.5$, $\theta=0$.

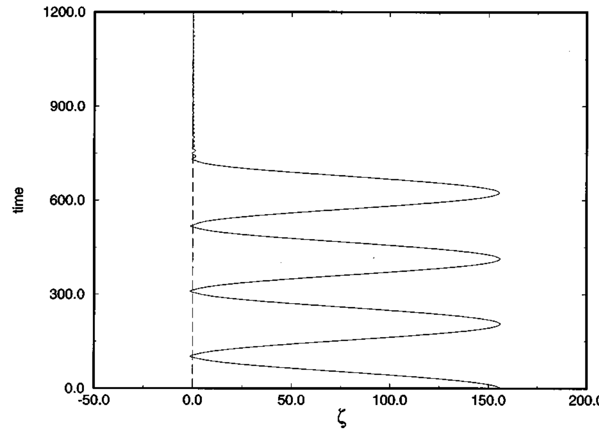

FIG. 3. Trajectory of a soliton with $w=0.5$, $\theta=0$, initially placed at $\zeta=156$, pinned by the impurity $0.26\delta_{n,0}$ after a number of the usual large amplitude oscillations in the presence of the linear potential $V_n=0.03n$. The dashed line indicates the location of the impurity.

ζ_0, and a soliton should be trapped at one of the ζ_0's at which $U''(\zeta_0)>0$. This is indeed the situation we observed in simulations. For example, for $\epsilon\kappa=0.14$, solitons (68) initially localized around $\zeta=0.5\sim1$ are trapped at $\zeta=1.37$. Note that for the above parameters, the two solutions for ζ_0 of Eq. (65) are $\zeta_0^{(1)}=0.4222$ at which $U''(\zeta_0)<0$, and $\zeta_0^{(2)}=1.353$, at which $U''(\zeta_0)>0$, respectively. The theoretical estimate for the trap position again is in good agreement with the numerical simulation. The relative error is less than 3%. Second, we performed simulations to verify the theoretical estimate of the trap threshold (67). For $w=0.5$ and $\gamma=0.015$, the theoretical estimate is $\epsilon_{\rm th}^{\rm theo}=0.103\,54$ with $\zeta_0=\pm0.8328$ at which $U''(\zeta_0)=0$. In the simulations, a soliton (68) was initially centered at $\zeta_0=0.8328$ or $\zeta_0=-0.8328$. For sufficiently strong impurities, it executed a small oscillatory motion (see Fig. 2). Numerically, we found there was a threshold $\epsilon_{\rm th}^{\rm exp}=0.1066$ below which no trapping was observed for $\zeta_0=0.8328$. The threshold for $\zeta_0=-0.8328$ was numerically found to be $\epsilon_{\rm th}^{\rm exp}=0.1013$. The thresholds $\epsilon_{\rm th}^{\rm theo}$ and $\epsilon_{\rm th}^{\rm exp}$ are within 3% relative error. Thus the theoretical estimate and the numerical results for the threshold are in excellent agreement.

(c) Finally, we point out that the trapping phenomenon is, in general, complicated. When a soliton will be trapped depends not only on the strength of the impurity, but also on its location, and the radiation induced by soliton interaction with the impurity. For example, in Fig. 3, we observe that a soliton of $w=0.5$, $\theta=0$, initially placed at $\zeta=156$ collides with the impurity $0.26\delta_{n,0}$ three times, accompanied by the emission of radiation (not shown), then it is finally trapped near the impurity at the fourth encounter. In numerical simulations, we recorded that the number of encounters before the final trapping may not be monotonically increasing with the decrease of the impurity strength. For instance, with the above set of parameters, we have a trapped soliton after eight encounters for $\kappa\epsilon=0.25$ and after six encounters for $\kappa\epsilon=0.27$ in contrast to the case shown in Fig. 3 which has four encounters for $\kappa\epsilon=0.26$.

IV. INTERACTION OF A SOLITON WITH DISSIPATIVE IMPURITIES

In this section, we focus on the soliton dynamics affected by the perturbation $R_n=-i\kappa\delta_{n,0}q_n$ with an imaginary κ: $\kappa=\pm i$, corresponding to a dissipative impurity. The equations of the adiabatic approximation now take the form

$$\frac{dw}{dt}=-i\frac{\epsilon\kappa}{2}\frac{\sinh^2(2w)}{\cosh[2w(1-\zeta)]\cosh[2w(1+\zeta)]},\qquad(69)$$

$$\frac{d\theta}{dt}=0,\qquad(70)$$

$$\frac{d\zeta}{dt}=-\frac{\sinh(2w)}{w}\sin[2(\Gamma+\theta)]+i\frac{\epsilon\kappa}{2w}\frac{\sinh^2(2w)\zeta}{\cosh[2w(1-\zeta)]\cosh[2w(1+\zeta)]},\qquad(71)$$

$$\frac{d\nu}{dt}=\gamma+2\cosh(2w)\cos[2(\Gamma+\theta)]+2\frac{\theta}{w}\sinh(2w)\sin[2(\Gamma+\theta)]-i\epsilon\kappa\frac{\theta}{w}\frac{\sinh^2(2w)\zeta}{\cosh[2w(1-\zeta)]\cosh[2w(1+\zeta)]}.\qquad(72)$$

As in Sec. III, the dynamics of ν is slaved to other parameters, and we confine ourselves to Eqs. (69) and (71) in the following. We notice that in terms of new variables $u=2w\zeta$, and $v=2w$ they can be rewritten as

$$\frac{dv}{dt}=\lambda\frac{\sinh^2 v}{\cosh(v-u)\cosh(v+u)},\qquad(73)$$

$$\frac{du}{dt}=-2\sinh v\sin\chi,\qquad(74)$$

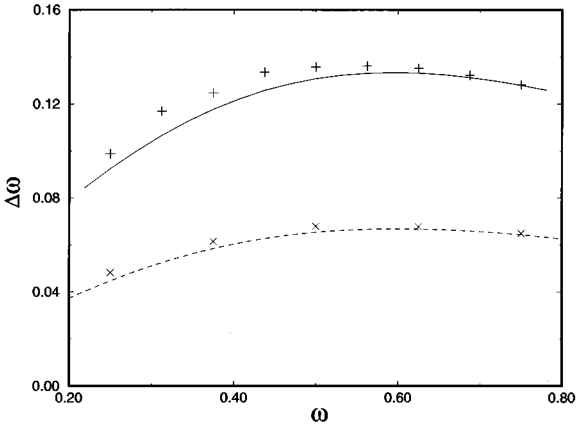

FIG. 4. Change of the amplitude parameter w for solitons of the initial profile [Eq. (68)] with $\theta=0.85$, $\zeta=0$ at $t=0$, interacting with a dissipative impurity $\kappa\epsilon\delta_{n,0}$, $\kappa=i$, $\epsilon=0.1$. (i) in the absence of a static ramp, i.e., $\gamma=0$, dashed line: the theoretical estimate [Eq. (76)]; crosses: numerical simulation results. (ii) in the presence of a static ramp, $\gamma=0.07$, solid line: from the theoretical estimate (78); plus: numerical simulation results. (See text for details.)

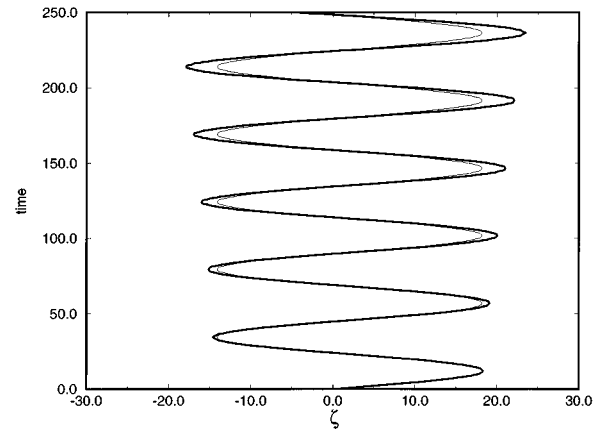

FIG. 5. Trajectories of a soliton with $w=0.4375$, $\theta=0.85$ initially localized at $\zeta=0$ interacting with a dissipative impurity $\kappa\epsilon\delta_{n,0}$ in the presence of a linear potential $V_n=0.14n$. Thick line: $\kappa\epsilon=0.1i$. For comparison, thin line: $\kappa\epsilon=0$, i.e., the impurity-free case.

where $\lambda=-i\epsilon\kappa$ is a small real parameter and $\chi=2\Gamma+2\theta$ (here we have taken into account the fact that θ is a constant).

A. Interaction of an AL soliton with a dissipative impurity

If $\gamma\equiv\Gamma\equiv0$, then $\chi=2\theta=\text{const}$ and Eqs. (73) and (74) reduce to

$$\frac{dv}{du}=\frac{\lambda}{2\sin\chi}\,\frac{\sinh v}{\cosh(v-u)\cosh(v+u)}. \tag{75}$$

A qualitative analysis of this equation gives rise to the following picture: The function v vs u is continuously decreasing (increasing) if $\lambda\sin\chi<0$ ($\lambda\sin\chi>0$). At infinity ($u\rightarrow\pm\infty$) v tends to a constant. This means that the interaction of a soliton with the impurity leads to decreasing (increasing) of the amplitude parameter w of the soliton if $\kappa=-i$ ($\kappa=i$). If we assume that the soliton comes from $\zeta=-\infty$ at $t=-\infty$ (consequently, $\sin\chi<0$), then the total change of the amplitude parameter w, i.e., $\Delta w=w(t=\infty)-w(t=-\infty)$ can be estimated as

$$\Delta w\approx-2i\epsilon\kappa\frac{w}{\sin2\theta\cosh2w}. \tag{76}$$

Here we have taken into account the fact that λ is a small parameter and hence the change Δw is small (i.e., it is assumed that $|\Delta w|\ll w$). Evidently, Eq. (76) remains valid also for a soliton moving from $+\infty$ toward the impurity (that is, when $\sin\chi>0$). Notice that the total norm of the system comprising only one soliton is $4w$ [see Eq. (52)]. Using this relation and assuming that the perturbed soliton retains the solitonic functional form (68), we can numerically estimate the parameter w. Of course, this procedure should produce an overestimate of w because the total norm of the system after the soliton passes the impurity consists of contributions from the soliton and the radiation it generates. The radiation constitutes the excess of w in numerical measurements. Bearing this in mind, we display in Fig. 4 the comparison between the theoretical estimate (76) and the results of numerical measurements. The discrepancy between theory and simulation ranges from 8% of relative errors for $w=0.25$ to 1% for $w=0.75$. This constitutes strong agreement.

B. Interaction of a soliton with the dissipative impurity in the presence of the constant field

Here we are concerned with the case of a static ramp field, i.e., $\gamma(t)=\gamma$, a constant. Since an unperturbed soliton executes periodic oscillations, we look for the averaged change of the characteristics during one period, as discussed above. Starting with the unperturbed solution,

$$u=\frac{1}{\gamma}\sinh v\cos(2\gamma t+2\theta)+u_0, \tag{77}$$

where $u_0=2w_0\zeta_0$, working within the first order of ϵ, and taking into account the fact that v changes slowly, we can represent the variation of the parameter v_0, $\Delta v_0=v_0(t+T)-v_0(t)$ during one period in the form of an integral,

$$\Delta v_0=2\lambda\sinh^2 v\int_t^{t+T}\frac{dt}{\cosh(2v)+\cosh(2u)}, \tag{78}$$

where one should substitute Eq. (77) into u, and regard v and u_0 as constants. The integral in Eq. (78) can be evaluated trivially in the case of small v, namely, $v\ll1$, and $v\ll\gamma$, to yield $\Delta v=\lambda Tv^2$. This estimate is replaced by $\Delta v=\lambda T\tanh^2 v$ in the limit of a strong field $\gamma\gg\sinh v$. To verify the theoretical estimate (78), we numerically integrated Eq. (78) and used the procedure mentioned above to carry out measurements for Δw in one oscillation of a soliton. The solid line shown in Fig. 4 is the result from the numerical integration of Eq. (78) (note that $\Delta w=\Delta v_0/2$, and $u_0=2w_0\zeta_0=0$ for the case shown), compared with the numerical measurements (pluses). As expected, there is an overestimate of Δw from the simulation. Clearly, the overall agreement is excellent, with relative errors ranging from less than 9% for $w \sim 0.25$ to less than 0.2% for $w \sim 0.75$. Figure 5 shows an example of the trajectories of solitons interacting with a dissipative impurity. It exhibits a qualitatively different behavior than the conservative case (cf. Fig. 1). The cumulative increase of the oscillation amplitude observed in the figure is a result of the incremental increase of the amplitude parameter w (for $\kappa = i$) since qualitatively the oscillation amplitude is proportional to $\sinh(2w)/w$ [see Eq. (57)].

V. CONCLUSIONS

In this work, we have developed a comprehensive perturbation theory for the inhomogeneous, discrete one-dimensional nonlinear Schrödinger equation (1) based on the inverse scattering transform. We have also discussed the adiabatic approximation and higher order corrections to this approximation for single-soliton dynamics. Using this formulation, we have discussed in detail the motion of a soliton interacting with a point impurity, either conservative or dissipative, in the presence of a spatially linear potential. We have predicted that there are two types of dynamical localization for the nondissipative impurity case, one being the usual dynamical localization, qualitatively the same as that in the absence of the impurity, the other being the pinning of a soliton by an impurity of sufficient strength. These predictions are confirmed by direct numerical simulations performed with the full dynamical system (1). Various theoretical estimates made within the adiabatic approximation are also shown to be in excellent agreement with numerical results.

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[1] *Nonlinear Coherent Structures in Physics and Biology*, edited by K.H. Spatchek and F.G. Mertens (Plenum, New York, 1994).

[2] *Future Directions of Nonlinear Dynamics in Physical & Biological Systems*, special issue of Physica D **68** (1993).

[3] D. Cai, A.R. Bishop, N. Grønbech-Jensen, and M. Salerno, Phys. Rev. Lett. **74**, 1186 (1995).

[4] V.V. Konotop, O.A. Chubykalo, and L. Vázquez, Phys. Rev. E **48**, 563 (1993)

[5] R. Scharf and A.R. Bishop, Phys. Rev. A **43**, 6535 (1991).

[6] M. Bruschi, D. Levi, and O. Ragnisco, Nuovo Cimento A **53**, 21 (1979).

[7] D.H. Dunlap and V.M. Kenkre, Phys. Lett. A **127**, 438 (1988); Phys. Rev. B **34**, 3625 (1986).

[8] V.V. Konotop, Phys. Rev. E **47**, 1473 (1993).

[9] V.V. Konotop, in *Fluctuation Phenomena: Disorder and Nonlinearity*, edited by A.R. Bishop, S. Jiménez, and L. Vázquez (World Scientific, Singapore, 1995), p. 258.

[10] V.I. Karpman, Phys. Scr. **20**, 462 (1979).

[11] A.A. Vakhnenko and Yu.B. Gaididei, Teor. Mat. Fiz. **68**, 350 (1986).

[12] M.J. Ablowitz and J.F. Ladik, J. Math. Phys. **16**, 598 (1975); **17**, 1011 (1976).

[13] A. Nayfeh, *Perturbation Methods* (Wiley, New York, 1973).

[14] V.V. Konotop, Teor. Mat. Fiz. **99**, 413 (1994).

[15] S. Homma and S. Takeno, Phys. Lett. A **169**, 355 (1992); S. Homma, in *Nonlinear Coherent Structures in Physics and Biology*, edited by. K.H. Spatchek and F.G. Mertens (Plenum, New York, 1994).